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A FINITENESS THEOREM FOR METRIC SPACES

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1. Introduction

We call a not necessarily continuous function $\rho\colon [0, R] \to [0, \infty)$ a contractibility function if $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\rho(\varepsilon) \geq \varepsilon$ for all ε. A metric space is called locally geometrically n-connected (resp. contractible) of size $\rho\colon [0, R] \to [0, \infty)$, if for all $x \in X$ the ball $B(x, r)$ is $0, 1, \cdots, n$-connected (resp. contractible) inside $B(x, \rho(r))$ for all $r \in [0, R]$. In symbols we merely write X is $\mathrm{LGC}^n(\rho)$ (resp. $\mathrm{LGC}(\rho)$). This property for metric spaces implies local n-connectedness (see [8]), but in general the notions are different.

Borsuk was the first to investigate the relation between local geometric contractibility and the topology of families of metric spaces (see [1]). More precisely he studied the metric and topological properties of families of subspaces of a metric space, under the constraint that these subspaces are $\mathrm{LGC}(\rho)$ for a fixed function ρ.

We shall see here that there is really not much difference in studying larger families of metric spaces. This very global point of view was promoted by Gromov (see e.g. [2]). The name LGC is also due to Gromov, and is justified by several applications in geometry (see e.g. [3]–[5]).

Define $\mathscr{M}(n, \rho)$ to be the collection of metric spaces that are $\mathrm{LGC}^n(\rho)$ and have covering dimension $\leq n$. The Hausdorff distance between metric spaces, as introduced by Gromov, induces a metric on $\mathscr{M}(n, \rho)$. Here the Hausdorff distance $H(X, Y) < \varepsilon$ iff there exists a metric d on $X \coprod Y$ inducing the original metrics on X and Y and so that

$$\max\{\sup\{d(x, Y)\colon x \in X\}, \sup\{d(y, X)\colon y \in Y\}\} < \varepsilon.$$

The object of this paper is to study the relation between this metric on $\mathscr{M}(n, \rho)$ and the homotopy types of spaces in $\mathscr{M}(n, \rho)$. The main result is

Theorem A. *There exists an $\varepsilon^*(n, \rho)$, depending only on n and ρ, so that if X, $Y \in \mathscr{M}(n, \rho)$ and $H(X, Y) < \varepsilon^*$, then X and Y are homotopy equivalent.*

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As an immediate corollary we get

Corollary B. *Let $\mathscr{C}$ be a precompact subset of $\mathscr{M}(n, \rho)$. Then $\mathscr{C}$ contains only finitely many homotopy types.*

It should be noted that ε^* in Theorem A can be computed directly, and so can the number of homotopy types in Corollary B provided the spaces are compact. It is also possible to bound the Betti numbers using the techniques developed here. For precise statements see §4.

Corollary B is an extension of a result by T. Yamaguchi.

Theorem [10]. *If $\mathscr{C}(n, R)$ is a precompact family of n-dimensional Riemannian manifolds, with criticality radius $\geq R$, then $\mathscr{C}$ contains only finitely many homotopy types.*

Here critically radius $\geq R$ implies that balls of radius $\leq R$ are contractible, so the spaces in question are in particular $\mathrm{LGC}^n(\mathrm{id}\colon [0, R] \to [0, R])$.

Corollary B is also an extension of a result by K. Grove and the author.

Theorem [7]. *Let $n \in \mathbf{N}$, $k \in \mathbf{R}$ and v, $D > 0$. The class of n-dimensional Riemannian manifolds with diameter $\leq D$, volume $\geq v$ and sectional curvatures $\geq k$ contains only finitely many homotopy types.*

Namely, it is possible from the proof to extract a uniform contractibility function for this class. The function will look like $\rho(\varepsilon) = C \cdot \varepsilon\colon [0, R] \to [0, \infty)$, where C, R depend on n, k, D, v.

The paper is organized as follows: In §2 there is a discussion on construction of maps from polytopes into $\mathrm{LGC}^n(\rho)$ spaces. The next section contains all the main results on existence of maps between Hausdorff close spaces and homotopies between close maps. In §4 the proof of Theorem A is presented together with a couple of results on how contractibility functions bound the topology.

In the last section we study convergence of compact $\mathrm{LGC}(\rho)$ spaces. Using results from [1] we show that if $\{X_n\}$ is a sequence of n-dimensional compact $\mathrm{LGC}(\rho)$ spaces converging to a compact space X, then X is also an n-dimensional $\mathrm{LGC}(\rho)$ space.

Because our spaces are not necessarily separable, we have chosen to work with the Hurewicz-Lebesgue covering dimension only.

The reader is referred to the two basic references [8], and [9] for notions and results not explicitly mentioned in the text.

2. Realization of polytopes

We show in particular that if P is an n-dimensional polytope and X is $\mathrm{LGC}^{n-1}(\rho)$, then any map from the vertices of P into X, where adjacent vertices are close in X, may be extended to a map from P into X.

For a contractibility function $\rho\colon [0, R] \to [0, \infty)$ define $\rho_1(\varepsilon) = \varepsilon + \rho(\varepsilon)$ and recursively $\rho_k(\varepsilon) = \varepsilon + \rho(\rho_{k-1}(\varepsilon))$. Notice that $\rho_k(\varepsilon) \to 0$ as $\varepsilon \to 0$ since $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Examples. If $\rho(\varepsilon) = \varepsilon$, then $\rho_k(\varepsilon) = (k+1)\varepsilon$. If $\rho(\varepsilon) = C \cdot \varepsilon$, then

$$\rho_k(\varepsilon) = \varepsilon + C \cdot \varepsilon + \cdots + C^k \cdot \varepsilon = \varepsilon \cdot \frac{C^{k+1} - 1}{C - 1}.$$

Assume P is a locally finite n-dimensional polytope and $Q \subset P$ is a subpolytope. For $0 \le k \le n$ we denote the k-skeleton of P by P^k, i.e. the union of simplicies of dimension $\le k$ in P. When $k+1$ vertices $v_0, \cdots, v_k$ span a simplex in P, we denote it by $\Delta(v_0, \cdots, v_k)$.

Main Lemma. *Let X be* $\mathrm{LGC}^{n-1}(\rho)$ *and $f\colon Q \cup P^0 \to X$ a continuous map with*

(i) *$d(f(v_0), f(v_1)) < \varepsilon$ for adjacent vertices in P^0,*

(ii) *if $\Delta \subset Q$ is a simplex, then* $\operatorname{diam} f(\Delta) < \varepsilon$.

If $\rho_{n-1}(\varepsilon) < R$ then f may be extended to a continuous map $\bar{f}\colon P \to X$, and if $\Delta \subset P$ is a simplex then $\bar{f}(\Delta) \subset B(f(v), \rho_n(\varepsilon))$ for any vertex v of Δ.

Remark. In case Q is finite, or more generally $f|Q$ is uniformly continuous, condition (ii) is superfluous, as we can just subdivide Q and thereby P until (ii) is satisfied and then extend f to the new vertices in $P - Q$ still keeping (i) true.

Proof. The construction is by induction on the skeleton of P.

Let $\Delta(v_0, v_1)$ be a 1-simplex. If $v_0, v_1 \in Q$ we are done, otherwise connect v_1 to v_0 by a curve in $B(f(v_0), \rho(\varepsilon))$. By construction $\bar{f}(\Delta[v_0, v_1]) \subset B(f(v_0), \rho(\varepsilon))$.

Next let $\Delta[v_0, \cdots, v_k]$ be a k-simplex. If $v_0, \cdots, v_k \in Q$ we are done, otherwise $\bar{f}(\partial\Delta[v_0, \cdots, v_k]) \subset B(f(v_0), \rho_{k-1}(\varepsilon))$. Whence $\bar{f}|\partial\Delta[v_0, \cdots, v_k]$ may be extended over $\Delta[v_0, \cdots, v_k]$ inside $B(f(v_0), \rho(\rho_{k-1}(\varepsilon)))$.

Because $\rho_1(\varepsilon) \le \rho_2(\varepsilon) \le \cdots \le \rho_{n-1}(\varepsilon)$, the procedure is possible as long as $\rho_{n-1}(\varepsilon) < R$. It furthermore follows from the construction that $\bar{f}(\Delta[v_0, \cdots, v_k]) \subset B(f(v_i), \rho_k(\varepsilon)) \subset B(f(v_i), \rho_n(\varepsilon))$, $i = 0, \cdots, k$. q.e.d.

Let X be $\mathrm{LGC}^n(\rho)$ and K an n-dimensional polytope.

Corollary. *Let f_0, $f_1\colon K \to X$ have the following properties:*

(i) *$d(f_0(k), f_1(k)) < \delta$ for all $k \in K$.*

(ii) $\operatorname{diam}(f_i(\Delta)) < \delta$, $i = 0, 1$, *for all simplicies in K.*

If $\rho_n(2\delta) < R$, then f_0 and f_1 are homotopic (here (ii) *is unnecessary again, provided K is finite or f_0, f_1 are uniformly continuous).*

Proof. Define $P = K \times [0,1]$ and $Q = K \times \{0,1\}$. The set Q is already triangulated, and using this, we can introduce a triangulation on P without adding any vertices. Now define $f: Q \to X$ as being f_i on $K \times \{i\}$, $i = 0, 1$. Then the conditions of the lemma are satisfied, if we set $\varepsilon = 2\delta$.

Remark. It is not trivial that close maps into a $\mathrm{LGC}^n(\rho)$ space are homotopic, because the contractions may not vary continuously from point to point.

3. Maps between close spaces

In this section we present the necessary material for the proof of Theorem A, starting with the construction of maps between Hausdorff close spaces.

Assume X is a metric space of dimension $\leq n$ and Y is $\mathrm{LGC}^{n-1}(\rho)$, where ρ is as above.

Proposition. *If $H(X,Y) < \varepsilon$ where $\rho_{n-1}(4\varepsilon) < R$, then there exists a continuous map $f: X \to Y$ with*

$$|d(x_1,x_2) - d(f(x_1), f(x_2))| < 4\varepsilon + 2\rho_n(4\varepsilon) \quad \textit{for all } x_1, x_2 \in X.$$

Proof. As X has dimension $\leq n$, it has an open covering α of order $\leq n+1$ and mesh $= \sup(\mathrm{diam}(U), U \in \alpha) < \varepsilon$. Choosing a partition of unity with respect to α we get a map $i: X \to N_\alpha$, where N_α is the nerve of α.

Next fix a metric d on $X \coprod Y$ inducing the original metrics on X and Y and so that X is contained in the ε-neighborhood around Y and vice versa.

Now choose for each $U \in \alpha$ a point $y_U \in Y$ with $d(U, y_U) < \varepsilon$. Since elements in α correspond to vertices in N_α, we get a map from the 0-skeleton of N_α into Y. Whenever U, $V \in \alpha$ are adjacent vertices in N_α, $d(y_U, y_V) < 4\varepsilon$ by the triangle inequality. We may therefore apply the lemma in §2 to get a map $f': N_\alpha \to Y$, provided $\rho_{n-1}(4\varepsilon) < R$. This gives us a map $f = f' \circ i: X \to Y$.

To estimate the "diameter" of $f: X \to Y$ let x_1, $x_2 \in X$. Choose $U_i \in \alpha$ with $x_i \in U_i$, $i = 1, 2$. Then $d(x_i, y_{U_i}) < 2\varepsilon$ and, by the construction of f',

$d(f(x_i), y_{U_i}) < \rho_n(4\varepsilon)$, $i = 1, 2$. Thus

$$\begin{aligned} d(f(x_1), f(x_2)) &\leq d(f(x_1), y_{U_1}) + d(y_{U_1}, y_{U_2}) + d(f(x_2), y_{U_2}) \\ &\leq 2\rho_{n-1}(4\varepsilon) + d(y_{U_1}, x_1) + d(y_{U_2}, x_2) + d(x_1, x_2) \\ &\leq 4\varepsilon + 2\rho_n(4\varepsilon) + d(x_1, x_2), \\ d(x_1, x_2) &\leq d(x_1, y_{U_1}) + d(y_{U_1}, y_{U_2}) + d(y_{U_2}, x_2) \\ &\leq 4\varepsilon + d(y_{U_1}, f(x_1)) + d(f(x_1), f(x_2)) + d(f(x_2), y_{U_2}) \\ &\leq 4\varepsilon + 2\rho_n(4\varepsilon) + d(f(x_1), f(x_2)). \end{aligned}$$

Remark. When the metric on $X \coprod Y$ is fixed, it follows that $d(x, f(x)) \leq 2\varepsilon + \rho_n(4\varepsilon)$ for all $x \in X$.

Remark. In the case Y sits inside X isometrically and X is contained in the ε-neighborhood of Y, one can, using a more careful construction, exhibit a retract $r\colon X \to Y$ as long as $\rho_n(24\varepsilon) < R$.

Remark. Everything in the previous proposition carries through just the same, if we only assume that there is a metric on $X \coprod Y$ inducing the original metrics on X and Y and so that X is contained in the ε-neighborhood around Y. In other words instead of assuming that $H(X, Y) < \varepsilon$ we may just assume that X is within ε of Y.

Assume now X is a metric space of dimension $\leq n$, which is also an ANR, and that Y is $\mathrm{LGC}^n(\rho)$.

Proposition. *Let f_0, $f_1\colon X \to Y$ be maps with the following properties*:

(i) $d(f_0(x), f_1(x)) < \varepsilon$ for all $x \in X$,

(ii) there exists $\bar{\delta} > 0$ so that if $A \subset X$ has $\operatorname{diam} A < \bar{\delta}$, then $\operatorname{diam} f_i(A) < \varepsilon$, $i = 0, 1$.

Then f_0 is homotopic to f_1 as long as $\rho_n(2\varepsilon) < R$. (Of course (ii) is superfluous when X is compact or f_0, f_1 are uniformly continuous, also it is enough to assume that X is merely metrizable.)

Proof. In the case where X is triangulable, all we have to do is to find a sufficiently small triangulation and then apply the corollary in §2. When X is only an ANR, however, we must approximate X by dominating polytopes.

In Chapter IV of [8] it is proved that there exists a covering α of X of order $\leq n + 1$ and maps $i\colon X \to N_\alpha$, $r\colon N_\alpha \to X$ with the properties that $r \circ i$ is homotopic to id_X and $\operatorname{diam} r(\Delta) < \bar{\delta}$ for any simplex in N_α. The map $i\colon X \to N_\alpha$ just corresponds to a partition of unity with respect to α.

The maps $f_0 \circ r$, $f_1 \circ r\colon N_\alpha \to X$ satisfy the conditions of the corollary in §2 with $\delta = \varepsilon$. Thus $f_0 \circ r$ and $f_1 \circ r$ are homotopic and therefore $f_0 \circ r \circ i$ and $f_1 \circ r \circ i$ are homotopic. Whence f_0 and f_1 are homotopic as id_X is homotopic to $r \circ i$.

4. Close spaces are homotopy equivalent

The road is now paved for a proof of Theorem A.

Theorem. *Let X, Y be metric spaces which are* $\mathrm{LGC}^n(\rho)$ *and have dimension $\leq n$. If $H(X,Y) < \varepsilon$, where $\rho_n(18\varepsilon + 8\rho_n(4\varepsilon)) < R$, then X and Y are homotopy equivalent.*

Proof. From §3 we conclude the existence of maps $f\colon X \to Y$ and $g\colon Y \to X$ with "diameter" $\leq 4\varepsilon + 2\rho_n(4\varepsilon)$, because $\rho_{n-1}(4\varepsilon) < R$. Thus if $A \subset X$ has $\operatorname{diam} A < \varepsilon$, we get $\operatorname{diam} g \circ f(A) \leq 9\varepsilon + 4\rho_n(4\varepsilon)$, and $d(x, g \circ f(x)) \leq 4\varepsilon + 2\rho_n(4\varepsilon)$ by the first remark after the first proposition in §3. The spaces X, Y are also ANR's since they have dim $\leq n$ and are locally n-connected (see [8]). The maps id_X and $g \circ f$ are therefore homotopic as long as

$$\rho_n(2(9\varepsilon + 4\rho_n(4\varepsilon))) = \rho_n(18\varepsilon + 8\rho_n(4\varepsilon)) < R.$$

Similarly id_Y is homotopic to $f \circ g$. q.e.d.

Since $\rho_n(18\varepsilon + 8\rho_n(4\varepsilon)) \to 0$ as $\varepsilon \to 0$ we can find $\varepsilon^*(n, \rho)$ so that $\varepsilon < \varepsilon^*$ implies $\rho_n(18\varepsilon + 8\rho_n(4\varepsilon)) < R$. This finishes the proof of Theorem A.

Example. Assume for instance that we have two n-dimensional spaces X, Y where balls of radius $\leq R$ are contractible within themselves. Then we may take $\rho(\varepsilon) = \varepsilon$ on $[0, R]$. Thus X is homotopic to Y as long as $H(X,Y) < \varepsilon^* \simeq R/(32n^2)$.

If, however, we knew for some other reason that R-close maps were homotopic, then we would only need $\varepsilon^* = R/[8(n+2)]$. This is, for example, the case when X, Y are Riemannian manifolds with criticality radius $\geq R$. In this case Yamaguchi shows that if X, Y are $R/[25(n+1)]$ close then they are homotopic.

Remark. If attention is restricted to compact spaces, the above arguments will become a little simpler. We can also improve the estimates up to a factor 4 by choosing coverings and approximating polytopes carefully.

Let us now turn to the question of how to bound the topology. When working with manifolds it is often possible to determine how many metric balls it takes to cover the space. Let us therefore assume that X is a metric space which is $\mathrm{LGC}^n(\rho)$, has dimension $\leq n$, and X can be covered by $\leq N(\varepsilon)$ metric balls for each $\varepsilon > 0$. Now fix ε^* so that $\rho_n(18\varepsilon^* + 8\rho_n(4\varepsilon^*)) < R$.

If X is covered by $\leq N(\varepsilon^*/2)$ metric balls of radius $\varepsilon^*/2$, then there is a refinement of this cover α of order $(n+1)$ which has no more than $(n+1)(N(\varepsilon^*/2)$ elements in it (see [9]). Applying the previous construction to the case where $Y = X$ we get a map $r\colon N_\alpha \to X$ so that $r \circ i$ is homotopic to id_X, where $i\colon X \to N_\alpha$ corresponds to a partition of unity. Hence N_α is a dominating polytope for X with no more than $(n+1)N(\varepsilon^*/2)$ vertices. In particular we get

Corollary.

$$\sum_{i=1}^{n} b_i(X,F) \le \sum_{k=0}^{n} \binom{(n+1)N(\varepsilon^*/2)}{k} \le (n+1)^{n+1}\left(N\left(\frac{\varepsilon^*}{2}\right)\right)^n,$$

where $b_i(X,F)$ is the ith Betti number with field coefficients F.

Example. Assume X is an n-dimensional Riemannian manifold with diameter $\le D$ and Ricci curvature ≥ 0. From the Bishop-Gromov Theorem (see [2]) M can be covered by $\le 2^n D^n \varepsilon^{-n}$ metric balls of radius ε. If furthermore the criticality radius of X is $\ge R$, it follows $\sum b_i(X,F) \le C(n,D,R)$, where $C(n,D,R) \simeq (D/R)^{n^2 \log n}$.

Let now $\mathscr{C}(n,\rho)$ be a precompact family of compact metric space which are $\mathrm{LGC}^n(\rho)$ and have dimension $\le n$. In [2] it is proved that for any precompact family there is a function $N(\varepsilon)$ so that any compact space in this family can be covered by $\le N(\varepsilon)$ ε-balls. Fix such a function $N(\varepsilon)$ for $\mathscr{C}(n,\rho)$. By the above discussion

$$\sum b_i(X,F) \le (n+1)^{(n+1)}\left(N\left(\frac{\varepsilon^*}{2}\right)\right)^n$$

for all $X \in \mathscr{C}$.

Also the number of homotopy types can be bounded. Following an argument by T. Yamaguchi in [10] one can see that $\mathscr{C}$ contains less than $(N(\varepsilon^*/2))^4$ homotopy types.

5. Convergence of LGC(ρ) spaces

If $\rho\colon [0,R] \to [0,\infty)$ is a contractibility function it is possible to find a concave contractibility function $\hat{\rho}\colon [0,R] \to [0,\infty)$ pointwise bigger than ρ (see [1]). Therefore, we may and shall in this section assume that our contractibility functions are concave.

Denote by $\mathscr{CM}$ the space of all compact metric spaces and $\mathscr{CM}(n,\rho)$ the subspaces of LGC(ρ) spaces of covering dimension $\le n$.

Theorem. *$\mathscr{CM}(n,\rho)$ is a closed subspace of $\mathscr{CM}$ with respect to the Hausdorff distance.*

Proof. Let X_k be a sequence in $\mathscr{CM}(n,\rho)$ converging to a compact metric space X.

To see that X has covering dimension $\leq n$ let $A \subset X$ be a finite-dimensional subspace. Using the third remark after the first proposition in §3, we get continuous maps $f_k\colon A \to X_k$ and a sequence $\varepsilon_k \to 0$ with $\operatorname{diam} f_k^{-1}(x) \leq \varepsilon_k$ for all $x \in X_k$, for k sufficiently large. Whence A has dimension $\leq n$, by Alexandorff's approximation theorem (see [9]). Thus all finite-dimensional subsets of X have dimension $\leq n$, but then X must have dimension $\leq n$ (see [9]).

Denote by $\bar{X}$ the disjoint union of X and X_k, $k = 1, 2, 3, \cdots$. In [6] it is proved that $\bar{X}$ can be equipped with a metric inducing the original metrics on X, X_k, $k = 1, 2, 3, \cdots$, and so that $X_k \to X$ in the classical Hausdorff metric on the subsets of $\bar{X}$. The space $\bar{X}$ is clearly compact and has dimension $\leq n$ since it is a countable union of closed spaces of dimension $\leq n$ (see [9]).

We can then apply the main theorem in §16 of [1] to see that X is $\mathrm{LGC}(\rho)$. q.e.d.

We can now sharpen Corollary B.

Corollary. *Let $\mathscr{C}$ be a precompact subset of $\mathscr{CM}(n, \rho)$. Then the closure $\bar{\mathscr{C}}$ of $\mathscr{C}$ in $\mathscr{CM}$ is a subset of $\mathscr{CM}(n, \rho)$, and contains only finitely many homotopy types.*

Corollary. *If $\{X_k\}$ is a sequence in $\mathscr{CM}(n, \rho)$ converging to a compact metric space X, then X is homotopy equivalent to X_k if k is sufficiently big.*

I am grateful to S. Ferry for pointing out the relevance of Corollary 3.2 in his paper "The homeomorphism group of a compact Hilbert cube manifold is an ANR," Ann. of Math. **106** (1977), 101–119. Using this, "homotopy type" may be replaced by "simple homotopy type" in the last two corollaries.

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